Existence of Balanced Simplices on Polytopes[1]

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Abstract

The classic Sperner lemma states that in a simplicial subdivision of a simplex in $\mathbb{R}^n$ and a labelling rule satisfying some boundary condition there is a completely labeled simplex. In this paper we first generalize the concept of completely labeled simplex to the concept of a balanced simplex. Using this latter concept we then present a general combinatorial theorem, saying that that under rather mild boundary conditions on a given labelling function there exists a balanced simplex for any given simplicial subdivision of a polytope. This theorem implies the well-known lemmas of Sperner, Scarf, Shapley, and Garcia as well as some other results as special cases. An even more general result is obtained when the boundary conditions on the labelling function are not required to hold. This latter result includes several results of Freund and Yamamoto as special cases.

Key words: Combinatorial theorems, integer labeling, fixed points, simplicial subdivision.

1 Introduction

The Sperner lemma (1928) is probably one of the elegant and fundamental results in combinatorial topology. It has become quite familiar in the fields of mathematical programming and economic equilibrium theory, because of its successful use in the computation of fixed points of a continuous function, see e.g. Scarf (1967, 1973), Kuhn (1968), Eaves (1972), Merrill (1972), van der Laan and Talman (1979), and many others. Surveys of the developments of the Sperner lemma can be found in Todd (1976), Forster (1980), Doup (1988) and Yang (1999). The lemma states that given a simplicial subdivision of the unit simplex

$$S^n = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i = 1\},$$

where $\mathbb{R}^n_+$ is the nonnegative orthant of the n-dimensional Euclidean space, and a labeling function L from the set of vertices of simplices of the simplicial subdivision into the set $\{1, \cdots, n\}$, satisfying that for any vertex x in the boundary of S^n that $L(x) \neq i$ when $x_i = 0$, there exists a completely labeled simplex, i.e. a simplex whose vertices carry all of the labels from 1 up to n. The Scarf lemma (1967, 1973) states a similar result when the labeling function satisfies that $L(x) = \min\{j \mid x_j = 0 \text{ and } x_{j+1} > 0\}$, with the convention that $n + 1 = 1$, when x is a vertex in the boundary of S^n. The Scarf lemma can be seen as a dual version of Sperner lemma and vice versa. However, the conditions in these two lemmas appear to be quite different. As far as we know, there is no result which has unified both the Sperner lemma and the Scarf lemma. The existing results extend either the scope of the Sperner lemma or that of the Scarf lemma.

In Cohen (1967) a stronger version of the Sperner lemma is given, which claims the existence of an odd number of completely labeled simplices. In Le Van (1982) an alternative proof of this result using topological degree theory is given. Shapley (1973) generalized the Sperner lemma by using a set labeling rule instead of an integer labeling rule. Furthermore, the existence of completely labeled simplices have been generalized to the cube and the simplotope, i.e. the Cartesian product of several simplices, while also more general labeling rules have been considered, see e.g. Tucker (1946), Fan (1967), Garcia (1976), van der Laan and Talman (1981, 1982), Freund (1984, 1986), van der Laan, Talman and Van der Heyden (1987), and Yamamoto (1988). In Freund (1989) the lemmas of Sperner, Scarf, and Garcia on a full-dimensional simplex are extended to a full-dimensional polytope, see also Yamamoto (1988). In Bapat (1989) a permutation-based generalization of the Sperner lemma has been presented.

In this paper we generalize the concept of completely labeled simplex to the concept of balanced simplex. A rather boundary condition on the labelling rule is formulated to guarantee the existence of a balanced simplex in any simplicial subdivision of a given polytope in $\mathbb{R}^n$. This leads to the first main theorem which implies most results mentioned above, including the lemmas of Sperner, Scarf, Shapley, and Garcia, as special cases and therefore unifies the Sperner and Scarf lemma. Secondly, allowing for more general labelings, we establish our second main theorem which unifies several results of Freund (1989) and Yamamoto (1988).

In Section 2 we discuss the basic notations and concepts related to polytopes and simplicial subdivisions. In Section 3 we present the first main theorem and illustrate the strength of the theorem by showing that it contains many well-known results as special cases. In Section 4 we present the second main theorem. Again many known results are a special cases of this theorem.

2 Preliminaries

For a convex set $B \subset \mathbb{R}^n$, let $\mathrm{bnd}(B)$, $\mathrm{int}(B)$ and $\dim(B)$ denote the relative boundary, the relative interior and the dimension of B, respectively. For k a nonnegative integer, the set of integers $\{1, \ldots, k\}$ is denoted by I_k, with the convention that $I_0 = \emptyset$. Given an integer k, $1 \leq k \leq n$, let be given a k-dimensional polytope P in $\mathbb{R}^n$. Then there exists an integer $m \geq k+1$, a set I of m integers, real vectors $a^i \in \mathbb{R}^n$, $i \in I$ and $d^h \in \mathbb{R}^n$, $h \in I_{n-k}$, and real numbers α_i, $i \in I$ and $\delta^h \in \mathbb{R}^n$, $h \in I_{n-k}$, such that P can be written as

$$P = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq \alpha_i,\ i \in I \text{ and } d^{h\top} x = \delta_h,\ h \in I_{n-k}\},$$

and chosen in such a way that none of the inequalities is an implicit equality and that none of the constraints is redundant. Given a subset B of P, we define the carrier of B as

$$\mathrm{Car}(B) = \{i \in I \mid a^{i\top} x = \alpha_i \text{ for all } x \in B\}.$$

For given polytope P, we define the set V by

$$V = \{x \in \mathbb{R}^n \mid x = \sum_{i \in I_{n-k}} \nu_h d^h,\ \nu_h \in \mathbb{R}\},$$

as the set of vectors spanned by a^i corresponding to the equality constraints, with $V = \{\underline{0}\}$ when $k = n$, and we define the set

$$V^* = \{x \in \mathbb{R}^n \mid x^\top y = 0 \text{ for all } y \in V\}$$

as the k-dimensional subspace orthogonal to V. For $T \subset I$, we further define

$$F(T) = \{x \in P \mid a^{i\top} x = \alpha_i \text{ for } i \in T\},$$

with $F(\emptyset) = P$. When $F(T)$ is nonempty, we call $F(T)$ a face of P. A face is called proper when the dimension of the face is at most equal to $k-1$ and a face $F(T)$ is called a vertex of P if the dimension of the face is zero. Finally, for $T \subset I$, we define

$$A(T) = \{x \in \mathbb{R}^n \mid x = \sum_{i \in T} \lambda_i a^i,\ \lambda_i \geq 0\} + V.$$

Observe that in case $k = n$ the set $A(T)$ is a cone spanned by the vectors a^i, $i \in T$, with top the zero vector $\underline{0}$.

Next, for given integer q, $0 \leq q \leq n$, a q-dimensional simplex or q-simplex in $\mathbb{R}^n$, denoted by $\sigma(x^1, \ldots, x^{q+1})$, in short by σ, is defined as the convex hull of $q + 1$ affinely independent vectors x^1, …, x^{q+1} in $\mathbb{R}^n$. For ℓ, $0 \leq \ell \leq q$, an ℓ-simplex being the convex hull of $\ell + 1$ vertices of σ is a face of σ. A finite collection $\mathcal{G}$ of k-simplices is a simplicial subdivision of the k-dimensional polytope P if

(a) P is the union of all simplices in $\mathcal{G}$;

(b) the intersection of any two simplices in $\mathcal{G}$ is either empty or a common face of both.

In the following $\mathcal{G}^+$ denotes the collection of all simplices in $\mathcal{G}$ and their faces and $\mathcal{G}^0$ denotes the the set of all vertices of the simplices in $\mathcal{G}$. When $\mathcal{G}$ is a simplicial subdivision of P, then for every face $F(T)$ of P the collection of all faces of $\mathcal{G}^+$ lying in $F(T)$ form a simplicial subdivision of $F(T)$. The simplicial subdivision of $F(T)$ induced by $\mathcal{G}$ is denoted by $\mathcal{G}(T)$, i.e.

$$\mathcal{G}(T) = \{\tau \subset F(T) \mid \tau = \sigma \cap F(T),\ \sigma \in \mathcal{G},\ \dim(\tau) = \dim(F(T))\}.$$

To introduce the concept of labeling function, let be given some arbitrary finite set J of at least $n + 1$ elements, called the labels, and a collection of vectors $c^j \in \mathbb{R}^n$, $j \in J$. For a nonempty set $S \subset J$, we define

$$C(S) = \mathrm{Conv}(\{c^j \mid j \in T\}),$$

where for $X \subset \mathbb{R}^n$, $\mathrm{Conv}(X)$ denotes the convex hull of X. A labeling function assigns an index from the set J to any vertex in the set $\mathcal{G}^0$. Let $L\colon \mathcal{G}^0 \to J$ be such a labeling rule and for a q-face $\sigma(x^1, \ldots, x^{q+1})$ in $\mathcal{G}^+$, let $L(\sigma) = \{L(x^1), \ldots, L(x^{q+1})\}$ denote the set of labels of the vertices of σ. We are now ready to define the concept of balanced simplices. It should be noticed that the balancedness of a simplex depends on the set J of labels and the collection c^j, $j \in J$, of vectors.

Definition 2.1
Let $\mathcal{G}$ be a simplicial subdivision of a polytope P. For given label set J and vectors c^j, $j \in J$, a q-simplex $\sigma(x^1, \ldots, x^{q+1})$ in $\mathcal{G}^+$ is balanced if $\underline{0} \in C(L(\sigma))$.

With slightly abuse of notation, we also call the collection $\{c^j \mid j \in L(\sigma)\}$ and the labelset $L(\sigma)$ balanced, when σ is balanced. More general, a set $S \subseteq J$ of labels is called balanced if $\underline{0} \in C(S)$, i.e. if the system of equations $\sum_{j \in S} \mu_j c^j = \underline{0}$ has a nonnegative solution satisfying $\sum_{j \in S} \mu_j = 1$. In the next section we formulate a sufficient condition to guarantee the existence of a balanced simplex in $\mathcal{G}^+$.

3 The existence of a balanced simplex

In this section we state the first main combinatorial theorem to be discussed in this paper. We further illustrate the strength and generality of the theorem by showing that a wide variety of combinatorial results appear to be a special case of the theorem. The theorem states a sufficient condition for existence of at least one balanced simplex in $\mathcal{G}^+$ for a given simplicial subdivision $\mathcal{G}$ of P.

Theorem 3.1 (Main Theorem I)
Let be given a k-dimensional polytope P in $\mathbb{R}^n$, $k \leq n$, a simplicial subdivision $\mathcal{G}$ of P, a finite nonempty set J of labels and a collection of vectors $\{c^j | j \in J\}$ in $\mathbb{R}^n$, satisfying $C(J) \cap V = \underline{0}$. Further, let $L: \mathcal{G}^0 \to J$ be a labeling rule such that for every simplex σ of the induced simplicial subdivision $\mathcal{G}(T)$ of a proper face $F(T)$ of P, the set $A(T) \cap C(L(\sigma))$ either is empty or contains the point $\underline{0}$. Then there exists a balanced simplex in $\mathcal{G}^+$.

Proof.
Let x be any point in P and let $\sigma(x^1, \ldots, x^{q+1})$ be the unique simplex in $\mathcal{G}^+$ containing x in its relative interior. Then there exist unique positive numbers $\gamma_1, \ldots, \gamma_{q+1}$ satisfying $\sum_{i=1}^{q+1} \gamma_i = 1$ such that $x = \sum_{i=1}^{q+1} \gamma_i x^i$. Then, let $f: P \to \mathbb{R}^n$ be a function defined at $x \in P$ by

$$f(x) = \sum_{i=1}^{q+1} \gamma_i c^{i_j},$$

where $i_j = L(x^j)$, $j = 1, \ldots, q+1$. Clearly, f is a continuous function from P to $C(J)$. Since P is compact and convex and f is continuous there exists an $x^* \in P$ being a stationary point of f on P, i.e.

$$x^\top f(x^*) \leq x^{*\top} f(x^*) \text{ for all } x \in P.$$

Consequently, x^* is a solution of the linear programming problem

$$\text{maximize } x^\top f(x^*) \text{ subject to } a^{i\top} x \leq \alpha_i,\ i \in I \text{ and } d^{h\top} x = \delta_h,\ h \in I_{n-k}.$$

Let $T^* \subset I$ be defined by $T^* = \{i \in I \mid a^{i\top} x^* = \alpha_i\}$. So, by definition $x^* \in F(T^*)$. Moreover, according to the duality theory in linear programming there exist $\lambda_i^* \geq 0$, $i \in T^*$ and $\nu_h^* \in \mathbb{R}$ for $h \in I_{n-k}$, such that

$$f(x^*) = \sum_{i \in T^*} \lambda_i^* a^i + \sum_{h=1}^{n-k} \nu_h^* d^h$$

and thus $f(x^*) \in A(T^*)$.

Next, let σ^* be any simplex of the induced simplicial subdivision $\mathcal{G}(T^*)$ of the face $F(T^*)$ of P containing x^*. Since $x^* \in \sigma^*$, we have $f(x^*) \in C(L(\sigma^*))$ and so $f(x^*) \in A(T^*) \cap C(L(\sigma^*))$. First, suppose that $T^* \neq \emptyset$. Then $F(T^*)$ is a proper face of P and therefore according to the boundary condition we have $\underline{0} \in A(T^*) \cap C(L(\sigma^*))$. Consequently, σ^* is balanced. Second, suppose that $T^* = \emptyset$ and thus $F(T^*) = P$. Then $A(T^*) = V$ and therefore $f(x^*) \in V \cap C(L(\sigma^*))$. Since $V \cap C(L(\sigma^*)) \subset V \cap C(J) = \underline{0}$ by the conditions of the theorem, it follows that $f(x^*) = \underline{0}$ and thus σ^* is balanced. Q.E.D.

A labeling rule L on $\mathcal{G}^0$ satisfying the boundary condition of the theorem is called a proper labeling rule. Furthermore, notice that the condition $C(J) \cap V = \underline{0}$ is satisfied if $\underline{0} \in C(J)$ and $C(J) \subset V^*$. Although a balanced simplex is not required to be of dimension k, it holds that every simplex of $\mathcal{G}$ containing a balanced simplex as a face is also balanced and hence the theorem says that when $C(J) \cap V = \underline{0}$ and the boundary condition holds the simplicial subdivision contains a k-dimensional balanced simplex. Here, it should be noticed that in all existing results in the literature, the boundary condition is imposed on every vertex of the simplicial subdivision lying on the boundary of the polytope. The novelty of Theorem 3.1 lies in the fact that the boundary condition is imposed on every simplex of the simplicial subdivision lying on the boundary of the polytope. The next result considers the case that the boundary condition is not required to hold and follows immediately from the proof of Theorem 3.1.

Corollary 3.2

For a finite collection of vectors $c^j \in \mathbb{R}^n$, $j \in J\}$, let $\mathcal{G}$ be a simplicial subdivision of the polytope P and let $L: \mathcal{G}^0 \to J$ be a labeling rule. Then there exist a set $T \subset I$ and a simplex $\sigma \in \mathcal{G}(T)$ with $A(T) \cap C(L(\sigma)) \neq \emptyset$.

To illustrate the strength of Theorem 3.1 we first consider several applications on the $(n-1)$-dimensional unit simplex S^n. For $h \in I_n$, S^n_h denotes the facet $S^n_h = \{x \in S^n \mid x_h = 0\}$, and for a proper subset $T \subset I_n$, $S^n(T) = \cap_{h \in T} S^n_h$. Furthermore, for $K \subset I_n$, let the n-vector m^K be defined by $\sum_{i \in K} \frac{1}{|K|} e^i$, where $|K|$ denotes the number of elements in K and e^i is the i-th unit vector in R^n. Observe that $m^K = e^i$ if $K = \{i\}$. For ease of notation we write $m^{I_n} = m$. Now, take $k = n-1$, $d^1 = m$, $\delta_1 = 1/n$, $m = k+1 = n$ and $I = I_n$, $a^i = m - e^i$ and $\alpha_i = 1/n$ for $i \in I$. Observe that $a^i \in V^*$ for all $i \in I$. For $K \subset I$, define $A'(K) = \{x \in \mathbb{R}^n \mid x = \sum_{i \in K} \lambda_i a^i,\ \lambda_i \geq 0, i \in K\}$. Now, the unit simplex S^n can be rewritten in the framework of this paper as

$$S^n = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq \alpha_i,\ i \in I \text{ and } d^{1\top} x = \delta_1\}.$$

We first apply Theorem 3.1 to prove the Sperner lemma (1928).

Theorem 3.3 (Sperner lemma)
Let $\mathcal{G}$ be a simplicial subdivision of S^n and let $L: \mathcal{G}^0 \to I_n$ be a labeling rule such that $L(x) \neq i$ when $x_i = 0$. Then there exists a completely labeled simplex of $\mathcal{G}$, i.e. a simplex $\sigma \in \mathcal{G}$ such that $L(\sigma) = I_n$.

Proof.
Take $J = I = I_n$ and for $j \in J$, set $c^j = a^{j+1}$. Clearly, $\underline{0} \in C(J)$ and $C(J) \subset V^*$. Therefore we have $C(J) \cap V = \{\underline{0}\}$. Notice that $\underline{0} \in C(K)$ if and only if $K = J$ and hence a balanced simplex must be full-dimensional and its vertices bear all labels 1 up to n. To show the existence of a balanced simplex it remains to show that the boundary condition of Theorem 3.1 is satisfied by every simplex in a proper face $S^n(T)$ of S^n. So, let $\sigma \in \mathcal{G}(T)$ for some nonempty $T \subset I$. Then $L(\sigma) \cap T = \emptyset$ since for every vertex x of σ we have $x_i = 0$ for every $i \in T$ and hence $L(x) \notin T$. Since the vectors a^i, $i \in S$, are linearly independent for any proper subset S of J we must have that $A'(L(\sigma)) \cap A(T) = \{\underline{0}\}$ and hence $C(L(\sigma)) \cap A(T) = \emptyset$. This completes the proof. Q.E.D.

Also the Scarf lemma (1967) can be proved by applying Theorem 3.1.

Theorem 3.4 (Scarf lemma)
Let $\mathcal{G}$ be a simplicial subdivision of S^n and let $L : \mathcal{G}^0 \mapsto I_n$ be a labeling rule satisfying $L(x) = \min\{i \mid x_i = 0$ and $x_{i+1} > 0\}$ for any vertex $x \in bnd(S^n)$ with the convention that $i + 1 = 1$ if $i = n$. Then there exists a completely labeled simplex of $\mathcal{G}$.

Proof.
Let $J = I_n$ and $c^j = -a^j$ for all $j \in J$. Again, $C(J) \subset V^*$ and $\underline{0} \in C(K)$ if and only if $K = J$. Hence a balanced simplex is full-dimensional and must carry all labels. It remains to prove that the boundary conditions of Theorem 3.1 are fulfilled for every simplex $\sigma \in \mathcal{G}(T)$ in any proper face $S^n(T)$. Suppose that $A(T) \cap C(L(\sigma)) \neq \emptyset$ for some nonempty subset T of J and some $\sigma \in \mathcal{G}(T)$. Then there exist nonnegative λ_i for $i \in T$, a real number ν_1, and nonnegative μ_j for $j \in S$ where $S = L(\sigma)$ such that $\sum_{i \in T} \lambda_i a^i + \nu_1 m = \sum_{j \in S} \mu_j c^j$ and $\sum_{j \in S} \mu_j = 1$. Since $c^j = -a^j$ for all $j \in J$, this yields

$$\sum_{i \in T} \lambda_i a^i + \sum_{j \in S} \mu_j a^j = -\nu_1 m.$$

Since $m^\top a^i = 0$ for all $i \in S \cup T$, it implies that $\nu_1 = 0$. It means that the vectors a^j, $j \in S \cup T$, are linearly dependent. Hence, $S \cup T = I_n = I = J$. Let $x^1, \cdots, x^{q+1}$ be the vertices of σ. Suppose that for some $j \in I_n$ it holds that $x^h_j > 0$ for all $h = 1, \ldots, q + 1$. Then $L(x^h) \neq j$ for all $h = 1, \ldots, q + 1$ and so $j \notin S$. Moreover, $j \notin T$. This contradicts the fact that $T \cup S = I_n$. Consequently, for every $j \in I_n$ there is at least one

$h \in \{1, \ldots, h+1\}$ satisfying $x_j^h = 0$. Since $T \neq I_n$ there is an $i \in I_n$ such that $i \notin T$ and $i+1 \in T$. Because $\sigma \in \mathcal{G}(T)$ there is an h with $x_i^h > 0$. Moreover, $i \notin S$ because of the fact that no vertex x^h can carry label i if $x_{i+1}^h = 0$. Hence, $i \notin T \cup S$, yielding a contradiction. Therefore, the conditions of Theorem 3.1 are satisfied and there exists a balanced simplex σ in $\mathcal{G}$ which must then be completely labeled. Q.E.D.

Notice that the properness condition in the Scarf lemma can be relaxed slightly. It is sufficient to require that $A(T) \cap C(L(\sigma)) = \emptyset$ for every simplex σ of $\mathcal{G}(T)$. The Theorems 3.3 and 3.4 show that both the Sperner lemma and the Scarf lemma are special cases of Main Theorem I. It is well-known that with respect to the boundary conditions the Scarf lemma can be seen as dual to the Sperner lemma. However, we are not aware of any other theorem containing both lemmas as special cases. This shows the generality of our result.

The next theorem to be proved by applying Theorem 3.1 was established in Shapley (1973). In this theorem the vertices of a simplicial subdivision of S^n are labeled with nonempty subsets of the set I_n. To prove the Shapley lemma, we need the concept of balancedness of sets. Let $\mathcal{N}$ be the collection of all nonempty subsets of the set I_n. A collection $\{B_1, \ldots, B_k\}$ of k elements of $\mathcal{N}$ is called balanced if the system of equations

$$\sum_{j=1}^{k} \lambda_j m^{B_j} = m$$

has a nonnegative solution.

Theorem 3.5 (Shapley lemma)
Let $\mathcal{G}$ be a simplicial subdivision of S^n and let $L\colon \mathcal{G}^0 \rightarrow \mathcal{N}$ be a labeling rule such that $L(x) \subset \{i \mid x_i > 0\}$ for any vertex $x \in S^n$. Then there exists at least one face $\sigma(x^1, \ldots, x^{q+1})$ of a simplex of $\mathcal{G}$ such that the collection $\{\, L(x^1), \ldots, L(x^{q+1}) \,\}$ is balanced.

Proof.
Let $J = \mathcal{N}$ and take $c^K = m - m^K$ for all $K \in \mathcal{N}$. Clearly, $C(J) \subset V^*$ and $\underline{0} \in C(J)$. We next prove that the boundary condition of Theorem 3.1 is satisfied by every simplex $\sigma(x^1, \cdots, x^{q+1})$ of $\mathcal{G}(T)$ for any nonempty subset T of I_n. Since $\sigma \in \mathcal{G}(T)$, we must have $x_j^i = 0$ for every $j \in T$, and hence according to the boundary condition $L(x^i) \cap T = \emptyset$ for all $i = 1, \ldots, q+1$. Let $B_i = L(x^i)$ for $i = 1, \ldots, q+1$ and $S = \cup_{i=1}^{q+1} B_i$. Then also $S \cap T = \emptyset$. Since the vectors a^i, $i \in K$, are linearly independent for each proper subset K of I_n we have that $A'(S) \cap A(T) = \{\underline{0}\}$. For every $i \in \{1, \ldots, q+1\}$ we have $L(x^i) \subset S$ and c^{B_i} is a convex combination of the vectors a^j, $j \in B_i$. Hence, $C(L(\sigma)) \subset A'(S)$. Moreover, since for every $i \in \{\, 1, \cdots, q+1 \,\}$ we have $c_j^{B_i} > 0$ for any $j \in T$, it implies that $\underline{0} \notin C(L(\sigma))$. Consequently, $C(L(\sigma)) \bigcap A(T) = \emptyset$ and hence the boundary condition is satisfied. This

guarantees the existence of a balanced simplex according to Theorem 3.1. Q.E.D.

The next result due to Garcia (1976) is a special case of Corollary 3.2. In this theorem no restriction is imposed on the labeling rule.

Theorem 3.6

Let $\mathcal{G}$ be a simplicial subdivision of S^n and let $L: \mathcal{G}^0 \to I_n$ be a labeling rule. Then there exists a simplex $\sigma \in \mathcal{G}^+$ such that $L(\sigma) \cup Car(\sigma) = I_n$.

Proof.

Let $J = I_n$ and let $c^j = -a^j$ for each $j \in J$. According to Corollary 3.2, there exists a simplex $\sigma \in \mathcal{G}(T)$ for some proper subset T of I_n such that $A(T) \cap C(L(\sigma)) \neq \emptyset$. Hence, the system of equations

$$\sum_{i \in T} \mu_i a^i + \beta m + \sum_{j \in L(\sigma)} \nu_j a^j = \underline{0}$$

has a solution $\mu_i^* \geq 0$, $i \in T$, β^*, and $\nu_j^* \geq 0$, $j \in L(\sigma)$ satisfying $\sum_{j \in L(\sigma)} \nu_j^* = 1$. Clearly the system has a solution only if $T \cup L(\sigma) = I_n$. Moreover, $T = \mathrm{Car}(\sigma)$. Hence $\mathrm{Car}(\sigma) \cup L(\sigma) = I_n$. Q.E.D.

We remark that the Sperner lemma, the Scarf lemma and the Garcia lemma have been generalized to the Cartesian product of unit simplices, see Freund (1986) and van der Laan and Talman (1982, 1987). It should be noticed that these generalizations can also be derived easily from Theorem 3.1. We want to conclude this section by stating some results on the n-dimensional unit cube $C^n = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1,\ i \in I_n\}$. Let $-I_n = \{-i \mid i \in I_n\}$. Notice that the cube can be seen as the Cartesian product of n one-dimensional unit simplices. The following lemmas on the cube are due to Freund (1984, 1986) and van der Laan and Talman (1981). Both lemmas say that under some condition on the labelling rule there exist in any simplicial subdivision $\mathcal{G}$ of C^n a complementary one-dimensional simplex, i.e. $\mathcal{G}^+$ contains an 1-simplex σ such that $L(\sigma) = \{k,\ -k\}$ for some $k \in I_n$. The proofs are omitted, but follow again immediately from applying Theorem 3.1.

Lemma 3.7

Let $\mathcal{G}$ be a simplicial subdivision of C^n and let $L: \mathcal{G}^0 \to I_n \cup -I_n$ be a labeling rule satisfying for every $x \in \mathcal{G}^0$ that $L(x) \neq i$ when $x_i = 1$ and $L(x) \neq -i$ when $x_i = 0$. Then $\mathcal{G}^+$ contains at least one complementary 1-simplex.

Lemma 3.8

Let $\mathcal{G}$ be a simplicial subdivision of C^n and let $L: \mathcal{G}^0 \to I_n \cup -I_n$ be a labeling rule such *that for every $x \in \mathcal{G}^0 \cap bnd(C^n)$ holds that $L(x) = i$ implies $x_i = 1$ and $L(x) = -i$ implies $x_i = 0$. Then $\mathcal{G}^+$ contains at least one complementary 1-simplex.*

The results discussed above show that Theorem 3.1. contains a wide variety of well-known combinatorial results as special cases and therefore illustrate the weakness of the conditions stated in Main Theorem I. In fact, a weak boundary condition together with $\underline{0} \in C(J)$ and $C(J) \subset V^*$ is enough. Remark that $V = \{\underline{0}\}$ when $k = n$. So, when P is a full-dimensional polytope, $V^* = \mathbb{R}^n$ and the boundary condition together with $\underline{0} \in C(J)$ is sufficient.

4 A combinatorial theorem on full-dimensional polytopes

The second main result of this paper is restricted to a full-dimensional polytope P in $\mathbb{R}^n$. So, the polytope is given by a system of $m \geq n+1$ inequalities, i.e. $k = n$ and the set I of m integers can be chosen to be $I = I_m$. To state the theorem, it should be noticed that it is always possible to take some arbitrarily chosen point $x^0 \in \mathrm{int}(P)$ and to scale the vectors a^i, $i \in I$ in such a way that P can be written as

$$P = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq 1 + a^{i\top} x^0, i \in I\}.$$

In the following a polytope P in this representation is said to be a polytope in standard form. Further we define $X = \mathrm{Conv}(\{a^j \mid j \in I\})$. Observe that if $F(T)$ is a face of P for some $T \subset I$, then the set $\mathrm{Conv}(\{a^j \mid j \in T\})$ is a face of X, see Grunbaum (1967), pp. 47-49. Given a nonempty label set J and a collection of vectors $c^j \in \mathbb{R}^n$, $j \in J$ we define for $y \in X$ the set $E(y) \subset J \times I$ by

$$\begin{aligned} E(y) \;\; = \;\; & \{(S,T) \subset J \times I \mid \exists \mu_j \geq 0,\; j \in S \text{ and } \mu_i,\; i \in T,\; \text{such that} \\ & \textstyle\sum_{j\in S} \mu_j c^j + \sum_{i \in T} \nu_i a^i = y \text{ and } \sum_{j \in S} \mu_j + \sum_{i\in T} \nu_i = 1\}. \end{aligned}$$

We now present the second main result, which says that for any nonempty set J of labels and correspondig vectors c^j, $j \in J$, any simplicial subdivision $\mathcal{G}$ of P, any labeling rule L and any element $y^0 \in X$, there is a simplex σ in $\mathcal{G}^+$ such that y^0 lies in the convex hull of the vectors c^j, $j \in L(\sigma)$ and a^i, $i \in \mathrm{Car}(\sigma)$.

Theorem 4.1 (Main Theorem II)
Let P be a polytope in standard form and for a nonempty finite set J, let $\{c^j \mid j \in J\}$ be a collection of vectors in $\mathbb{R}^n$. Let $\mathcal{G}$ be a simplicial subdivision of the n-dimensional polytope P and let $L\colon \mathcal{G}^0 \to J$ be a labeling rule. Then for each $y^0 \in int(X)$, there exists a simplex $\sigma \in \mathcal{G}^+$ such that $(L(\sigma),\; Car(\sigma)) \in E(y^0)$.

Proof.

Let x be any point in P and let $\sigma(x^1, \ldots, x^{q+1})$ be the unique simplex in $\mathcal{G}^+$ containing x in its relative interior. Then there exist unique positive numbers γ_1, $\ldots$, γ_{q+1} satisfying $\sum_{i=1}^{q+1} \gamma_i = 1$ such that $x = \sum_{i=1}^{q+1} \gamma_i x^i$. For given $y^0 \in \text{int}(X)$, define the correspondence $\xi\colon P \rightarrow \mathbb{R}^n$ by

$$\xi(x) = \text{Conv}(\{y^0 - c^j \mid j = L(x^i) \text{ if } \gamma_i = \max_h \gamma_h \}).$$

Now, consider the polytope

$$Q = \{x \in \mathbb{R}^n \mid {a^i}^\top x \leq 2 + {a^i}^\top x^0,\ i \in I\},$$

containing P in its interior. For a point $x \in Q \setminus P$, let λ_x be the unique number in $(0, 1)$ such that $x^0 + \lambda_x(x - x^0) \in \text{bnd}(P)$ and define $p(x) = x^0 + \lambda_x(x - x^0)$. Now we define the correspondence $\psi\colon Q \rightarrow \mathbb{R}^n$ by

$$\psi(x) = \begin{cases} \xi(x), & \text{if } x \in \text{int}(P) \\ \text{Conv}(\xi(x) \cup \{y^0 - a^i \mid i \in \text{Car}(x)\}), & \text{if } x \in \text{bnd}(P) \\ \text{Conv}(\{y^0 - a^i \mid i \in \text{Car}(p(x))\}), & \text{if } x \in Q \setminus P. \end{cases}$$

The correspondence ψ is upper semi-continuous, nonempty-valued, convex-valued and compact-valued. For a compact convex set Y containing $\cup_{x \in Q} \psi(x)$, let $\phi\colon Y \rightarrow Q$ be a correspondence, defined by

$$\phi(y) = \{x \in Q \mid z^\top y \leq x^\top y \text{ for all } z \in Q\}.$$

The correspondence ϕ is upper semi-continuous, nonempty-valued, convex-valued and compact-valued. Hence $\psi \times \phi\colon Y \times Q \rightarrow Y \times Q$, defined by $(\phi \times \psi)(y, x) = \phi(y) \times \psi(x)$, is upper semi-continuous, nonempty-valued, convex-valued, and compact-valued. So, according to Kakutani's fixed point theorem there exists a pair of vectors $(y^*, x^*) \in Y \times Q$ such that $y^* \in \psi(x^*)$ and $x^* \in \phi(y^*)$. The latter implies that

$$z^\top y^* \leq {x^*}^\top y^* \text{ for all } z \in Q.$$

Consequently, x^* is a solution of the linear programming problem

$$\text{maximize } z^\top y^* \text{ subject to } {a^i}^\top z \leq 2 + {a^i}^\top x^0,\ i \in I.$$

We now show that $x^* \in P$. Therefore, let $T^* = \{i \in I \mid {a^i}^\top x^* = 2 + {a^i}^\top x^0\}$. According to the duality theory in linear programming there exist real numbers $\lambda_i^* \geq 0$ for $i \in T^*$, such that $y^* = \sum_{i \in T^*} \lambda_i^* a^i$.

First, suppose $T^* \neq \emptyset$. Then $x^* \in \text{bnd}(Q)$ and thus $\psi(x^*) = \text{Conv}(\{y^0 - a^i \mid i \in \text{Car}(p(x^*))$. Since $x^* \in \text{bnd}(Q)$ we have that $\lambda^*_x = \frac{1}{2}$ and it follows that $\text{Car}(p(x^*)) = T^*$. So, there exist nonnegative numbers μ^*_i, $i \in T^*$, summing to one such that

$$\sum_{i \in T^*} \mu^*_i (y^0 - a^i) = y^* = \sum_{i \in T^*} \lambda^*_i a^i.$$

Hence $y^0 = \sum_{i \in T^*} (\mu^*_i + \lambda^*_i) a^i$ with $\sum_{i \in T^*} (\mu^*_i + \lambda^*_i) \geq 1$, contradicting $y^0 \in int(X)$. So, we must have that $T^* = \emptyset$ and thus $y^* = \sum_{i \in \emptyset} \lambda^*_i a^i = \underline{0}$.

Second, suppose x^* lies in the interior of Q but not in P. Then, it follows from $y^* \in \psi(x^*)$ that $y^* = \sum_{i \in \text{Car}(p(x^*))} \mu^*_i (y^0 - a^i) = \underline{0}$ for some nonnegative numbers μ^*_i with $\sum_{i \in \text{Car}(p(x^*))} \mu^*_i = 1$. So, $y^0 = \sum_{i \in \text{Car}(p(x^*))} \mu^*_i a^i$, contradicting that $y^0 \in \text{int}(X)$ and $F(\text{Car}(p(x^*)))$ is a face of P. So, $x^* \in P$.

To complete the proof, we consider the next two cases. First, suppose $x^* \in \text{int}(P)$ and thus $y^* \in \xi(x^*)$. Then there is a unique simplex σ with $\text{Car}(\sigma) = \emptyset$ containing x^* in its interior. Let $w^1, \ldots, w^{t+1}$ be the vertices of σ. Then by definition of $\xi(x^*)$ there exist nonnegative numbers μ^*_j, $j \in L(\sigma)$, with sum equal to one such that $\sum_{j \in L(\sigma)} \mu^*_j (y^0 - c^j) = y^* = \underline{0}$. So, $y^0 \in \text{Conv}(\{c^j \mid j \in L(\sigma)\})$ and thus $(L(\sigma),\ \text{Car}(\sigma)) \in E(y^0)$. Second, suppose $x^* \in \text{bnd}(P)$. Then $y^* \in \text{Conv}(\xi(x^*) \cup \{y^0 - a^i \mid i \in \text{Car}(x^*)\})$. Then there is a unique simplex σ with $\text{Car}(\sigma) = \text{Car}(x^*)$ containing x^* in its interior. Let $w^1, \ldots, w^{t+1}$ be the vertices of σ. Then we have

$$\sum_{j \in L(\sigma)} \mu^*_j (y^0 - c^j) + \sum_{i \in \text{Car}(\sigma)} \nu^*_i (y^0 - a^i) = y^* = \underline{0}$$

for some nonnegative numbers μ^*_j, $j \in L(\sigma)$, ν^*_i, $i \in \text{Car}(\sigma)$, with

$$\sum_{j \in L(\sigma)} \mu^*_j + \sum_{i \in \text{Car}(\sigma)} \nu^*_i = 1.$$

Hence, $(L(\sigma),\ \text{Car}(\sigma)) \in E(y^0)$. Q.E.D.

We show the generality of the theorem by discussing three results of Freund (1989) on an arbitrarily given full-dimensional polytope defined by

$$P = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq 1, i \in I\}$$

with $|I| \geq n+1$. Since by definition P is bounded, the point $\underline{0}$ lies in the convex hull of the vectors a^i, $i \in I$. Also, $V = \{\underline{0}\}$. Recall that the n-dimensional set X denotes the convex hull of the vectors a^i, $i \in I$, with $\text{Conv}(\{a^i \mid i \in T\})$ a face of X when $F(T)$ is a face of P. For $y \in X$, we define $D(y) = \{T \subset I \mid y \in \text{Conv}(\{a^j \mid j \in T\})\}$, i.e. $D(y)$ is the collection of all sets $T \subset I$ satisfying that $y \in \text{Conv}(\{a^j \mid j \in T\})$. Let $\mathcal{G}$ be a simplicial subdivision of P. A simplicial subdivision $\mathcal{G}$ of P is called bridgeless if for each $\sigma \in \mathcal{G}$, the intersection of all faces of P that meet σ is nonempty. In the following results the set J of labels is taken to be equal to the set I. For given simplicial subdivision $\mathcal{G}$, a labeling rule $L: \mathcal{G}^0 \to I$ is called dual proper if $L(x) \in \mathrm{Car}(x)$ for all $x \in \mathrm{bnd}(P)$.

The first theorem to be stated is a generalization of Theorem 3.6 from the simplex to a full-dimensional polytope. The proof is omitted, because it follows easily from applying Theorem 4.1 by taking $J = I$ and $c^j = a^j$ for all $j \in J$. It should be noticed that Theorem 17 of Yamamoto (1988) is a special case of the theorem.

Theorem 4.2
Let $\mathcal{G}$ be a simplicial subdivision of P and let $L: \mathcal{G}^0 \to I$ be a labeling rule. Then for each $y \in int(X)$, there exists a simplex σ in $\mathcal{G}^+$ such that $Car(\sigma) \cup L(\sigma) \in D(y)$.

The next result generalizes Theorem 3.4 to the full-dimensional polytope and follows easily again from Theorem 4.1 by taking $J = I$ and $c^j = -a^j$ for all $j \in J$. It should be noticed that the boundary condition on the labelling rule in Theorem 3.4 guarantees that each label in the labelset I_n is carried by at least one of the vertices in $\mathcal{G}^0$, implying that each label occurs at least once. Otherwise, not all labels need to occur and then of course the theorem does not need to hold. In the next theorem the bridgeless condition together with the properness of the labelling rule guarantees the occurrence of enough different labels to obtain the result.

Theorem 4.3
Let $\mathcal{G}$ be a bridgeless simplicial subdivision of P and let $L: \mathcal{G}^0 \to I$ be a dual proper labeling rule. Then for each $y \in int(X)$ there exists a simplex σ in $\mathcal{G}^+$ such that $L(\sigma) \in D(y)$.

The last theorem extends Theorem 3.3 to the full-dimensional polytope and follows again easily from Theorem 4.1 by taking $J = I$ and $c^j = -a^j$ for all $j \in J$. Observe from the definition of $E(y)$ that in this case $E(y)$ is the collection of all subsets $S \times T$ of $I \times I$, such that y is in the convex hull of the vectors a^j, $j \in S \cup T$.

Theorem 4.4
Let $\mathcal{G}$ be a simplicial subdivision of P and let $L: \mathcal{G}^0 \to I$ be a labeling rule. Then for each $y \in int(X)$, there exists a simplex σ in $\mathcal{G}^+$ such that $(L(\sigma), Car(\sigma)) \in E(y)$.

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